Binomial expansion for non-commutative operators

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Theorem 1. *Let us operators a and b (such as creation and annihilation operators) such that $[a, b] = c$ and $[a, [a, b]] = [b, [a, b]] = 0$.*

Then,

$$(a+b)^n = \sum_{\substack{k=0 \\ n \equiv k \pmod 2}} \left(-\frac{c}{2}\right)^{\frac{n-k}{2}} \frac{n!}{k!(\frac{n-k}{2})!} \left(\sum_{r=0}^{k} \binom{k}{r} a^r b^{k-r}\right)$$

Proof. From Baker-Hausdorff lemma, we have

$$e^{t(a+b)} = e^{ta} e^{tb} e^{-ct^2/2}$$

Left hand side can be written

$$e^{t(a+b)} = \sum_k \frac{(a+b)^k t^k}{k!}$$

where one recognizes the term we want to compute.

Right hand side is given by

$$e^{ta} e^{tb} e^{-ct^2/2} = \sum_{i,j,k} t^{i+j+2k} \frac{a^i b^j (-c/2)^k}{i!\, j!\, k!}$$

Identifying both sides, one has:

$$\frac{(a+b)^n}{n!} = \sum_{i+j+2k=n} \frac{(-c/2)^k}{i!\, j!\, k!} a^i b^j$$

$$\frac{(a+b)^n}{n!} = \sum_{\substack{i+j \leq n \\ i+j=n[2]}} \frac{(-c/2)^{(n-i-j)/2}}{i!\, j!\, \left(\frac{n-i-j}{2}\right)!} a^i b^j$$

Let us note $k = m + n$ and $r = m$,

$$\frac{(a+b)^n}{n!} = \sum_{\substack{0\le r\le k\le n\\ k=n[2]}} \frac{(-c/2)^{(n-k)/2}}{\left(\frac{n-k}{2}\right)!\, r!\,(k-r)!} a^r b^{k-r}$$

$$\frac{(a+b)^n}{n!} = \sum_{\substack{k=0\\ k=n[2]}}^{n} \frac{(-c/2)^{(n-k)/2}}{k!\left(\frac{n-k}{2}\right)!} \sum_{r=0}^{k} \binom{k}{r} a^r b^{k-r}$$

□

Based on https://mathoverflow.net/questions/78813/binomial-expansion-for-non-commutative-setting/.